Module-2: Bilinear Transformation - Basic Properties

Another important class of elementary mappings was studied by Augustus Ferdinand Möbius. These mappings are conveniently expressed as the quotient of two linear expressions and is defined as follows.

Definition 1. *If a, b, c, d are complex constants then the transformation*

$$w = f(z) = \frac{az+b}{cz+d} \tag{1}$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$ *is called a Bilinear Transformation or Möbius Transformation or linear fractional transformation. The expression $ad - bc$ is called the determinant of the transformation.*

Note 1. *The transformation (1) can also be written as*

$$Azw + Bz + Cw + D = 0, \quad AD - BC \neq 0.$$

Since this is linear in both the variables z and w, (1) deserves to be termed bilinear transformation.

Remark 1. *When $c = 0$, (1) represents simply a linear transformation. When $c \neq 0$, then the transformation (1) can be written as*

$$\begin{aligned} w = \frac{az+b}{cz+d} &= \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\ &= \frac{a}{c} - \frac{ad-bc}{c} \cdot \frac{1}{cz+d}. \end{aligned}$$

If $ad - bc = 0$, then $w = \frac{a}{c}$ = constant. Thus the condition $ad - bc \neq 0$ means that the function $w = f(z)$ is non-constant.

Theorem 1. *The inverse of a bilinear transformation is also a bilinear transformation.*

Proof. Let

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

be a bilinear transformation. Solving for z we obtain from above

$$z = \frac{-dw+b}{cw-a}, \tag{2}$$

where the determinant of the transformation is $ad-bc$ which is not zero. Thus the inverse of a bilinear transformation is also a bilinear transformation. □

Remark 2. *From the bilinear transformation (1) and its inverse (2) it follows that to every z other than $z = -d/c$ (w has a simple pole at $z = -d/c$) there corresponds only one value of w and to every value of w other than $w = a/c$ (z has a simple pole at $w = a/c$) corresponds just one value of z. We suppose that the point at infinity in the w-plane corresponds to the point $z = -d/c$, and that the point at infinity in the z-plane is mapped into the point $w = a/c$. Thus if $c \neq 0$, $z = \infty$ corresponds to $w = a/c$ and $z = -d/c$ corresponds to $w = \infty$. When $c = 0$, the point $z = \infty$ corresponds to $w = \infty$. Therefore, for $c \neq 0$, we have*

$$w = f(z) = \begin{cases} \frac{az+b}{cz+d}, & \text{if } z \neq -d/c, \ z \neq \infty \\ \infty, & \text{if } z = -d/c \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$$

It now follows that the bilinear transformation (1) set up a one-one correspondence between the points of the extended z-plane and the points of the extended w-plane.

Theorem 2. *A bilinear transformation is a conformal mapping for all finite z except $z = -d/c$.*

Proof. Let $w = f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ be a bilinear transformation. Then

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0 \; for \; z \neq -d/c,$$

and so $w = f(z)$ is a conformal mapping for all finite z except $z = -d/c$. □

Theorem 3. *The composition of two bilinear transformation is again a bilinear transformation.*

Proof. Let

$$\begin{aligned} \zeta &= \frac{a_1 z + b_1}{c_1 z + d_1}, \quad a_1 d_1 - b_1 c_1 \neq 0 \\ \textit{and} \quad w &= \frac{a_2 \zeta + b_2}{c_2 \zeta + d_2}, \quad a_2 d_2 - b_2 c_2 \neq 0 \end{aligned}$$

be two bilinear transformations. Substituting we obtain

$$\begin{aligned} w &= \frac{a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2} = \frac{(a_1 a_2 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2) z + (b_1 c_2 + d_1 d_2)} \\ &= \frac{az + b}{cz + d}, \end{aligned}$$

where $a = a_1 a_2 + b_2 c_1$, $b = a_2 b_1 + b_2 d_1$, $c = a_1 c_2 + c_1 d_2$, $d = b_1 c_2 + d_1 d_2$. Again

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} \neq 0.$$

Thus, the composition of two bilinear transformation is again a bilinear transformation.

□

Theorem 4. *The identity mapping $w = z$ is a bilinear transformation.*

Proof. We have

$$w = z = \frac{1 \cdot z + 0}{0 \cdot z + 1},$$

which is obviously a bilinear transformation. □

Theorem 5. *The associative law for composition of bilinear transformation holds.*

Proof. Let

$$\begin{aligned} T_1 \, : \, \zeta &= \frac{a_1 z + b_1}{c_1 z + d_1}, \quad a_1 d_1 - b_1 c_1 \neq 0, \\ T_2 \, : \, \lambda &= \frac{a_2 \zeta + b_2}{c_2 \zeta + d_2}, \quad a_2 d_2 - b_2 c_2 \neq 0 \\ \textit{and} \ T_3 \, : \, w &= \frac{a_3 \lambda + b_3}{c_3 \lambda + d_3}, \quad a_3 d_3 - b_3 c_3 \neq 0 \end{aligned}$$

be three bilinear transformations. Then

$$\begin{aligned} T_2 T_1 \, : \, \lambda &= \frac{a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2} \\ &= \frac{(a_1 a_2 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2) z + (b_1 c_2 + d_1 d_2)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
T_3(T_2T_1)\ :\ w &= \frac{a_3\frac{(a_1a_2+b_2c_1)z+(a_2b_1+b_2d_1)}{(a_1c_2+c_1d_2)z+(b_1c_2+d_1d_2)}+b_3}{c_3\frac{(a_1a_2+b_2c_1)z+(a_2b_1+b_2d_1)}{(a_1c_2+c_1d_2)z+(b_1c_2+d_1d_2)}+d_3} \\
&= \frac{(a_1a_2a_3+a_3b_2c_1+a_1b_3c_2+b_3c_1d_2)z+(a_2a_3b_1+a_3b_2d_1+b_1b_3c_2+b_3d_1d_2)}{(a_1a_2c_3+b_2c_1c_3+a_1c_2d_3+c_1d_2d_3)z+(a_2b_1c_3+b_2c_3d_1+b_1c_2d_3+d_1d_2d_3)} \\
&= \frac{az+b}{cz+d},\quad say.
\end{aligned}$$

Again

$$\begin{aligned}
T_3T_2\ :\ w &= \frac{a_3\frac{a_2\zeta+b_2}{c_2\zeta+d_2}+b_3}{c_3\frac{a_2\zeta+b_2}{c_2\zeta+d_2}+d_3} \\
&= \frac{(a_2a_3+b_3c_2)\zeta+(a_3b_2+b_3d_2)}{(a_2c_3+c_2d_3)\zeta+(b_2c_3+d_2d_3)}.
\end{aligned}$$

So

$$\begin{aligned}
(T_3T_2)T_1\ :\ w &= \frac{(a_2a_3+b_3c_2)\frac{a_1z+b_1}{c_1z+d_1}+(a_3b_2+b_3d_2)}{(a_2c_3+c_2d_3)\frac{a_1z+b_1}{c_1z+d_1}+(b_2c_3+d_2d_3)} \\
&= \frac{(a_1a_2a_3+a_3b_2c_1+a_1b_3c_2+b_3c_1d_2)z+(a_2a_3b_1+a_3b_2d_1+b_1b_3c_2+b_3d_1d_2)}{(a_1a_2c_3+b_2c_1c_3+a_1c_2d_3+c_1d_2d_3)z+(a_2b_1c_3+b_2c_3d_1+b_1c_2d_3+d_1d_2d_3)} \\
&= \frac{az+b}{cz+d}.
\end{aligned}$$

This shows that $T_3(T_2T_1) = (T_3T_2)T_1$ and so the associative property holds for the composition of bilinear transformation. □

Example 1. *Show by an example that bilinear transformations are not commutative under composition.*

Solution. *We consider the bilinear transformations*

$$T_1(z) = \frac{z}{z+1};\quad T_2(z) = \frac{z-1}{z-2}.$$

Therefore,

$$\begin{aligned}
T_1T_2(z) &= T_1(T_2(z)) = T_1\left(\frac{z-1}{z-2}\right) = \frac{z-1}{2z-3}, \\
T_2T_1(z) &= T_2(T_1(z)) = T_2\left(\frac{z}{z+1}\right) = \frac{1}{z+2}.
\end{aligned}$$

Hence, $T_1T_2(z) \neq T_2T_1(z)$. This shows that bilinear transformations do not satisfy the commutative property.

Note 2. *The set of all bilinear transformations form a non-commutative group with respect to the composition of maps.*

Theorem 6. *Every bilinear transformation maps circles and lines into circles and lines (a line is a circle of infinite radius).*

Proof. Let $w = f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ be a bilinear transformation. If $c = 0$, then

$$f(z) = \frac{a}{d}z + \frac{b}{d} = Az + B, \quad A = \frac{a}{d} \text{ and } B = \frac{b}{d}.$$

Clearly, $Az + B$, being linear, maps circles and lines into circles and lines.

If $c \neq 0$, then

$$\begin{aligned} f(z) &= \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\ &= \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z + d/c}. \end{aligned}$$

Assigning

$$z_1 = z + d/c, \quad z_2 = \frac{1}{z_1}, \quad z_3 = \frac{bc-ad}{c^2} z_2$$

we obtain $f(z) = \frac{a}{c} + z_3$. It is clear that the above transformations are of the form

$$w_1 = z + \alpha, \quad w_2 = \frac{1}{z}, \quad w_3 = \beta z.$$

This establishes the fact that every bilinear transformation is the resultant of bilinear transformations with simple geometric imports. Thus, a bilinear transformation maps circles and lines into circles. This proves the theorem. □

Example 2. *Show that the transformation* $w = \frac{2z+3}{z-4}$ *maps the circle* $x^2 + y^2 - 4x = 0$ *onto the line* $4u + 3 = 0$.

Solution. *Given transformation is clearly a bilinear transformation. The inverse transformation is given by*

$$z = \frac{4w+3}{w-2}.$$

The equation of the circle can be written as

$$x^2 + y^2 - 4x = 0$$

$$i.e. \quad |z|^2 - 4Re\, z = 0$$

$$i.e. \quad z\overline{z} - 2(z + \overline{z}) = 0.$$

Putting the value of z *and noting that* $w = u + iv$, *we obtain from above*

$$\frac{4w+3}{w-2} \cdot \frac{4\overline{w}+3}{\overline{w}-2} - 2\left(\frac{4w+3}{w-2} + \frac{4\overline{w}+3}{\overline{w}-2}\right) = 0$$

$$i.e. \quad (4w+3)(4\overline{w}+3) - 2\{(4w+3)(\overline{w}-2) + (4\overline{w}+3)(w-2)\} = 0$$

$$i.e. \quad 2(w + \overline{w}) + 3 = 0$$

$$i.e. \quad 4u + 3 = 0,$$

which is the required line.

Example 3. *Show that the line* $x = 3y$ *is mapped onto the circle under the bilinear transformation* $w = \frac{iz+2}{4z+i}$. *Find the centre and radius of the image circle.*

Solution. *Given transformation is clearly a bilinear transformation. The inverse transformation is given by*

$$z = \frac{-iw+2}{4w-i}.$$

Putting $z = x + iy$ *and* $w = u + iv$ *we obtain*

$$\begin{aligned} x + iy &= \frac{(v+2) - iu}{4u + i(4v-1)} \\ &= \frac{[(v+2) - iu][4u - i(4v-1)]}{16u^2 + (4v-1)^2} \\ &= \frac{9u - i(4u^2 + 4v^2 + 7v - 2)}{16u^2 + (4v-1)^2}. \end{aligned}$$

Comparing real and imaginary parts we obtain

$$x = \frac{9u}{16u^2 + (4v-1)^2}, \quad y = -\frac{4u^2 + 4v^2 + 7v - 2}{16u^2 + (4v-1)^2}.$$

Putting these values in the equation $x = 3y$ *we get*

$$\begin{aligned} \frac{9u}{16u^2 + (4v-1)^2} &= -3\frac{4u^2 + 4v^2 + 7v - 2}{16u^2 + (4v-1)^2} \\ i.e. \quad u^2 + v^2 + \frac{3}{4}u + \frac{7}{4}v - \frac{1}{2} &= 0, \end{aligned}$$

which represents a circle with centre at $(-3/8, -7/8)$ *and radius* $\frac{3}{4}\sqrt{\frac{5}{2}}$.

Example 4. *Find the image of the annulus* $\{z : 1 <| z |< 2\}$ *under the bilinear transformation* $w = \frac{z}{1-z}$.

Solution. *Here the transformation is* $w = \frac{z}{1-z}$. *Solving for* z *we obtain*

$$z = \frac{w}{1+w}.$$

Now

$$| z | > 1 \Leftrightarrow | w |^2 >| 1 + w |^2, \quad i.e. \; 0 > 1 + 2Re\, w$$

$$\text{and} \quad | z | < 2 \Leftrightarrow | w |^2 < 4 | 1 + w |^2, \quad i.e. \; 0 < 3[| w + 4/3 |^2 - 4/9].$$

Therefore, it is easily seen that

(i) $| z |> 1$ *is mapped into* $Re\, w < -1/2$ *and*

(ii) $| z | < 2$ *is mapped onto* $| w + 4/3 | > 2/3$.

Thus the required image is $\{w : Re\, w < -1/2\} \cap \{w : | w + 4/3 | > 2/3\}$.